Some Problems in the Theory of Radiation.

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1. The following paper deals with the oscillatory energy taken up by a simple resonator under the action of white light, and the translatory energy imparted to a molecule by radiation.

The first problem has already been treated by Planck: it is solved here in a very simple manner, and the method used, when applied to the second problem, leads to the important result, that a molecule at rest within an enclosure of uniform temperature will, under the action of the radiations, be set in motion with an acceleration that will increase its speed until the average energy reaches a definite value. If the Rayleigh-Jeans laws of radiation be assumed to hold, the ultimate average energy due to radiation alone is two-thirds of that derived from the kinetic theory of gases.

2. The connexion between radiation and absorption is sometimes illustrated by means of a pendulum acted on by a succession of blows. If these be delivered always in the same direction, and at intervals equal to the period of free oscillation, the amplitude rapidly increases; and the pendulum may then be said to absorb the energy of the blows. When the intervals are irregular, the impulses varying in phase and direction, it is further argued that their effects will be neutralized, so that the pendulum remains at comparative rest. I used at one time to try and illustrate the distinction experimentally in the lecture-room, but met with no success. However much I delivered the blows indiscriminately and irregularly, the pendulum always showed a continuously increasing amplitude of oscillation. I ultimately abandoned the experiment, thinking that there was some unconscious psychological influence which forced the experimenter to time his blows—to some extent—in unison with the time of swing. I ought to have known better, for anyone who has given attention to the nature of white light should have seen at once that the continuously increasing oscillation of the pendulum set in motion by irregularly delivered blows, each imparting the same velocity irrespective of the phase, is the true analogy of what occurs in nature when white light is absorbed in a responsive medium. If there were no continuously increasing energy of motion there could be no absorption of white light.

The complete theory shows that with regular intervals equal to that of the free period, the *amplitude* increases continuously with the time, while, when the blows succeed each other irregularly, it is the average *energy* that is proportional to the time. In the absence of friction the energy increases indefinitely in both cases.

3. Let a particle perform simply periodic oscillations under the action of a force directed to a fixed centre, the velocity being represented by $p \cos n\,(t-\tau)$. If at the time t_1, an additional velocity q be imparted to the particle in its line of motion, the subsequent velocity will be represented by:

$$p \cos n\,(t-\tau)+q \cos n\,(t-t_1),$$

for it is easily seen that the last term introduces no change of position at time t_1, but increases the velocity by the quantity q. The result of the sudden change of velocity is therefore the same as if an additional oscillation $q \cos n\,(t-t_1)$ had been superposed on the original one. It follows that the effect of successive blows q_1, q_2, etc., delivered at times t_1, t_2, etc., may be obtained by taking the resultant of a number of oscillations determined by their maximum velocities q_1, q_2, and phases nt_1, nt_2, etc.

If the intensities of the blows be equal, and the phases distributed at random, it follows, from a well-known theorem of Lord Rayleigh, that the expectancy of the resultant intensity is $\mathrm{N}q^2t$, where N is the number of blows delivered in unit time. The measure of the intensity here is the square of the maximum velocity of the particle. Writing m for its mass, the expectancy of maximum kinetic energy is therefore $\frac{1}{2}m\mathrm{N}q^2t$, and the same expression may stand for the sum of the potential and kinetic energies in the resulting oscillations. When the motion of the particle is damped so that its average energy Q diminishes according to the exponential law, the decrease of energy per unit time, due to damping, will be proportional to Q say, λQ. A state will ultimately be reached where the diminution due to damping will be equal to the energy supplied by the impulses, and the expectancy of the energy will then be:

$$\mathrm{Q} = \mathrm{N}mq^2/2\lambda. \qquad (1)$$

The condition that the oscillator has only one degree of freedom and that the impulses are delivered in its line of motion may now be dropped. If we resolve the impulses and the velocity of the oscillator along three co-ordinate axes, the result applies to each co-ordinate separately, the average square of the component of the blows along one co-ordinate being equal to a third of the square of the resultant, and, adding up the three resulting intensities, the final result remains the same. It is understood, of course, that if the oscillator has only one degree of freedom, and the blows are delivered indiscriminately in all directions, we should only get one-third of the resultant intensity.

4. Before applying our results, a few words may be said on the method of treating white light as a series of impulses. This was first suggested by Gouy, and favourably commented on by Rayleigh, and I have used it with advantage, but its value has never been sufficiently recognized. A non-homogeneous radiation is completely defined by the distribution of energy, which —when resolved spectroscopically—is contained within a definite range of frequencies. Two sets of disturbances, however they may differ in detail, give, so far as our observational powers go, identical results so long as their energy distribution is the same. In the first consideration of the subject, light is naturally introduced as a periodic phenomenon, homogeneous radiations alone being dealt with. In some of the more complicated effects we are then led to consider white light as a superposition of homogeneous radiations, though we can never obtain a truly continuous spectrum in this manner. If we have recourse to Fourier's theorem, certain difficulties arise which may easily be overcome, but they are often best avoided by considering a non-homogeneous radiation to consist of a superposition of similar disturbances having a finite and preferably short duration. If it be required to represent completely a given radiation, each impulse must—when resolved spectroscopically—contain the required distribution of energy; but in the important series of problems in which the range of wave-length affecting the solution of the problem is small, this condition is not essential. Within the effective range, the energy may be looked upon as constant, and the remainder of the spectrum does not matter. For my purpose it is sufficient to consider the impulse to be confined to a narrow layer of thickness $c\tau$ along a plane progressive wave-front (c = velocity of light). Within the layer I take the electromotive force E to be constant. There is an equal magnetic force at right angles to E, and the energy transmitted in unit time per unit surface of the wave-front is $E^2c\tau/4\pi$. If there are N equal impulses following each other in unit time, the energy per unit volume is $NE^2\tau/4\pi$. It will be shown in § 10 that when the disturbance is analysed spectroscopically, the energy per unit volume and unit range of frequency is connected with the impulse $E\tau$ by the relation

$$U = NE^2\tau^2/2\pi. \qquad (2)$$

5. We are now prepared to write down the result for the ultimate average energy of an electric oscillator under the action of light impulses. The velocity imparted to the electron at each impulse is $Ee\tau/m$, and equation (1) therefore becomes

$$\begin{aligned} Q &= NE^2e^2\tau^2/2\lambda m, \\ &= 2\pi e^2U/\lambda m. \end{aligned}$$

λ represents the fractional loss of energy of the oscillator due to its radiation per unit time, and this is shown in the electromagnetic theory of light to be equal to $2e^2\omega^2/3mc^3$, $\omega/2\pi$ being the frequency. Hence

$$Q = 3\pi c^3 U/2\omega^2.$$

This agrees with Planck's solution of the problem, except in so far as his oscillator is assumed to have only one degree of freedom, which reduces the average energy to one-third of the above value. The Rayleigh-Jeans law of radiation may be put into the form

$$3\pi c^3 U = 4\omega^2 W;$$

where W represents the average kinetic energy of the molecule of a gas, and is proportional to the absolute temperature. This gives finally

$$Q = 2W. \tag{3}$$

As half the energy of the oscillator is potential, we conclude that the ultimate kinetic energy of an oscillator having three degrees of freedom is equal to the translatory energy of a molecule in accordance with the law of equi-partition. This has already been pointed out by Planck. The proposition is subject to the same limitations as the Rayleigh-Jeans law, and holds therefore only for low frequencies, or high temperatures, the condition being that the product of temperature and wave-length shall be large.

It is instructive to compare the above method of treatment with that employed by Planck. Both assume the intensity distribution to be uniform within the effective range of wave-length, and the result is therefore not affected by considering it to remain uniform throughout the spectrum. Planck, starting from homogeneous oscillations, has to calculate the result of each within the narrow limits of frequencies closely adjoining that of the free period of the oscillator. He thus obtains the resulting energy by integration. I avoid the integration by starting from the integrated light disturbance which, in the case of a disturbance having uniform intensity along a frequency spectrum, consists of a series of impulses within each of which the electric force is constant.

6. We may now deal with the mechanical effect on the oscillator as a whole which is due to the magnetic force carried by the impulse. If M be the mass of the molecule, and v the velocity of the oscillating electron at any time, its direction forming an angle θ with the magnetic force, the velocity generated by the impulse, considering that the magnetic and electric forces are numerically equal, is: $\mathrm{E}e\tau v \sin\theta/c\mathrm{M}$. We have to find the average square of the velocities for all possible combinations of θ and v. Collecting together the cases in which v has the same numerical value, we note that

the average square of $\sin^2\theta$ is $\frac{2}{3}$. The rate of increase of energy of the molecule, to which we attach a mass M, is therefore;

$$\tfrac{1}{3}c^{-2}M^{-1}Ne^2E^2\tau^2v^2 = \tfrac{1}{3}2\pi c^{-2}e^2Uv^2/M, \qquad (4)$$

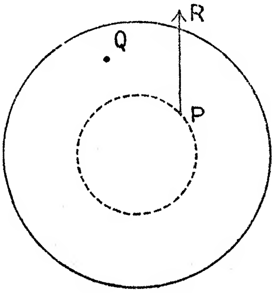
where v^2 now stands for the average square of the oscillatory velocity of the electron.

Using a result obtained by M. Abraham,* which will be further discussed in the next paragraph, it appears that a moving oscillator is acted on by a retarding force which is equal to $\frac{2}{3}\omega^2v^2e^2u/c^5$, if u be its translatory velocity. The diminution of energy per unit time is, therefore, equal to this expression multiplied by u. Equating the rate of increase and diminution of energy, we obtain for the ultimate state:

$$\begin{aligned} \omega^2Mu^2 &= \pi c^3U, \\ \tfrac{1}{2}Mu^2 &= \tfrac{1}{2}\pi c^3U\omega^{-2}, \\ &= \tfrac{2}{3}W = \tfrac{1}{3}Q. \end{aligned} \qquad (5)$$

The kinetic energy of the molecule tends, therefore, to become equal to two-thirds of that obtained by the kinetic theory of gases, and equal to one-third of the energy of internal motion.

7. The retarding force acting on the moving resonator is most easily determined by calculating the momentum transmitted to the enclosure; but the law of action and reaction is not applicable, because a momentum that leaves the oscillator in the direction of motion in time dt will reach the enclosure in time $dt(1-u/c)$. Generally, if the direction of transmission be defined by θ, the angle it forms with the direction of motion, the ratio of the rate at which the momentum is received to that at which it leaves the oscillator is $1-\beta\cos\theta$. The force on the enclosure is therefore $1-\beta\cos\theta$ times that acting on the oscillator. The difference is made up by an accumulation of momentum in the medium. That such an accumulation actually takes place may be shown directly. Consider a point P within a spherical enclosure radiating equally in all directions. At any point Q we may associate (according to Poynting) a momentum density with the energy density. If its value at unit distance from P be M, its value at Q will be M/r^2. If the angle between PQ and PR, the direction of motion, be θ, and we only consider the momentum transmitted along PR, we may confine ourselves to the effective part of the momentum density, which is $M\cos\theta/r^2$. To obtain its total value we must sum up over the whole sphere. It is clear



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* 'Elektromagnetische Theorie der Strahlung,' p. 116.

that, as the point P moves, the momentum towards R will diminish, and that in the opposite direction increase, hence there is an accumulation of momentum in the direction opposed to that of the motion. The total momentum transmitted in the direction PR is easily obtained if we note that the integral of $M \cos \theta / r^2$ is the same as that of the component, in the direction PR, of an attracting mass spread with uniform density M over the sphere, the law of action being that of the inverse square. This resultant force is then equal to the component resolved along PR of the attraction of a sphere, concentric with the enclosure and passing through P. If we take as axis of y a line passing through the centre parallel to PR, we find for the resultant momentum resolved along the axis of y the expression $-4\pi My/3$. The momentum M may now be expressed in terms of the energy of radiation. The energy density being Mc/r^2, the amount traversing a closed surface enclosing P in unit time is $4\pi Mc^2$, and if this be denoted by I, we find for the rate of increase of the momentum resolved in the direction of motion within the enclosure $\frac{1}{3}c^{-1}I\beta$. This must, therefore, be the excess of force acting on the enclosure over that acting on the radiant source.

8. We have now to determine the value of the force retarding the translatory velocity of a radiating body in virtue of its own radiation. Abraham has given the complete solution of this problem for an oscillator moving with a velocity which is great compared with that of the periodic motion (*loc. cit.*). This is not the case in the problem here considered, but it will be shown that his solution holds generally to the first order of magnitude in u/c. The subject being important, it is permissible to enter into more detail than would be necessary for our main object.

Abraham's equations give the electric and magnetic forces due to an electron possessing any given velocities and accelerations. It will be found convenient to separate at once the oscillatory velocity v from the translatory velocity u, and to write for the total velocity $u+v$, both quantities being looked upon as vectors. Abraham's vector-equations, by simple transformations, then become

$$\mathrm{E} = \mathrm{A}\,\{r_1(\dot{v}_1, r_1) - \dot{v}_1 - \beta u_1(\dot{v}_1, r_1) + \beta \dot{v}_1(r_1, u_1)\},$$

$$\mathrm{H} = \mathrm{A}\,\{[\dot{v}_1, r_1] + \beta(\dot{v}_1, r_1)[u_1, r_1] - \beta(r_1, u_1)[\dot{v}_1, r_1]\}.$$

Round brackets here denote scalar products and square brackets vector products. The suffix " 1 " denotes that the vector has unit length, so that $\dot{v}_1$ would simply indicate the direction of the acceleration, and $(\dot{v}_1, r_1)$ the cosine of the angle between the direction of the acceleration of the electron and the radius vector, drawn from the electron to the point at which the electric and magnetic forces are required. In the second equation it has been assumed that the velocity and acceleration are in the same direction, so that it can only be applied to linear oscillations. The quantity β denotes u/c. All terms of the equations (omitting the factor A) have zero dimensions, and may be expressed in terms of angular co-ordination. A has the value $ec^{-2}r^{-1}\dot{v}[1-c^{-1}((u+v)r_1)]^{-3}$. If we wish to express our results in rectangular co-ordinates, we must distinguish the case of an oscillation at right angles to the direction of motion u from that in which the oscillation is parallel to u. In the former case we obtain, if u be along the axis of y, and the oscillations along z,

$$\left.\begin{aligned} E_1 &= A\frac{xy}{r^2}, & H_1 &= -A\left(\frac{y}{r}-\beta\frac{y^2+z^2}{r^2}\right), \\ E_2 &= A\left(\frac{y^2}{r^2}-\beta\frac{z}{r}\right), & H_2 &= A\left(\frac{x}{r}-\beta\frac{xy}{r^2}\right), \\ E_3 &= -A\left(\frac{x^2+y^2}{r^2}-\beta\frac{y}{r}\right), & H_3 &= -A\beta\frac{xz}{r^2}. \end{aligned}\right\} \tag{6}$$

In the case of longitudinal oscillations, the terms in β disappear, and we have

$$\left.\begin{aligned} E_1 &= A\frac{xy}{r^2}, & H_1 &= A\frac{z}{r}, \\ E_2 &= -A\frac{x^2z^2}{r^2}, & H_2 &= 0, \\ E_3 &= A\frac{yz}{r^2}, & H_3 &= -A\frac{x}{r}. \end{aligned}\right\} \tag{7}$$

The energy density is $A^2\{(x^2+y^2)+\beta^2(y^2+z^2)-2\beta yr\}/4\pi r^2$ for transverse vibrations, and $A^2(x^2+z^2)/4\pi r^2$ for longitudinal vibrations.

The above equations hold without limit as regards the velocity, but, if we wish to proceed further, we have to examine the factor A, which contains a term $c^{-1}(v+u)r_1$, or $\beta\cos(r, u)+\beta'\cos(r, v)$ if $\beta' = v/c$. When $\beta = 0$, we return to the stationary oscillator, and β' introduces a small correction to the Hertzian equations. Its effect would—*inter alia*—be to introduce higher harmonics. But all effects of the first power of β' are periodic, averaging out during a complete oscillation, and for our purpose we may therefore neglect them altogether. Confining henceforward our investigation to terms not higher than the first power of β, we may put $c^2rA = (1+3\beta y/r)e\dot{v}$.

To obtain the rate at which the momentum leaves the oscillator, we have according to § 7, to multiply our expression by $(1-\beta\cos\theta)$. Taking all these matters into consideration, and introducing polar co-ordinates,

$x = r\sin\theta\sin\phi$, $y = r\cos\theta$, $z = r\sin\theta\cos\phi$, the total rate of momentum leaving the oscillator is

$$(4\pi c^4)^{-1}e^2\dot{v}^2\int_0^\pi\int_0^{2\pi}\{(\cos^2\theta+\sin^2\theta\sin^2\phi)(1+5\beta\cos\theta)-2\beta\cos\theta\}\cos\theta\sin\theta\,d\theta\,d\phi.$$

The term not affected by β disappears, when integrated over the sphere, and we ultimately find $\frac{2}{3}e^2\dot{v}^2c^{-4}\beta$ for the retarding force acting on the oscillator. As we need only consider average values, we may substitute ω^2v^2 for $\dot{v}^2$, and we thus obtain the result used in the previous article. For longitudinal vibrations, we obtain similarly for the rate of loss of momentum

$$(4\pi c^4)^{-1}e^2\dot{v}^2\int_0^\pi\int_0^{2\pi}\sin^2\theta\cos\theta\,(1+5\beta\cos\theta)\sin\theta\,d\theta\,d\phi.$$

This integral is found to have the same value as the previous one.

The rate of emission of energy by a stationary simple oscillator is $2e^2\dot{v}^2/3c^3$, and the retarding force is therefore $\mathrm{I}u/c^2$, independently of the direction of vibration. Hence the same expression holds for any particle, or, indeed, for a solid body emitting white light. By an altogether different method Lorentz arrived at the same result. Poynting, on the other hand, gives $\frac{2}{3}\mathrm{I}u/c^2$ for the retarding force in the body of his paper, and reduces that still further in a note at the end. The discrepancy lies in the value given to the intensity sent out by a moving radiating surface in a direction forming an angle θ with the direction of motion. According to Poynting, this is proportional to $1+2\beta\cos\theta$, while our equations, based mainly on Abraham's calculations, lead to $1+4\beta\cos\theta$.

9. We may now briefly cover the ground again, with a view to dismissing all unnecessary assumptions. The most important of these is, that the molecule acts as a simple oscillator, *i.e.*, emits homogeneous radiations. The calculation remains substantially the same, for an electron performing orbits round a nucleus. If the electric and magnetic forces of the incident radiation be concentrated in impulses, each impact adding a velocity V, the square of the velocity, which originally was v^2, will have increased to $v^2+\mathrm{V}^2+2v\mathrm{V}\cos\gamma$, where γ is the angle between V and v. The angle γ being arbitrary, the average increase of the square of the velocity will be V^2. The rate of increase of internal energy is therefore the same as that previously found: $\mathrm{NE}e^2\tau^2/2m$. But we cannot, without knowing the laws of intra-molecular action, decide what fraction of that increased energy is kinetic, because the value of an instantaneous increment of velocity does not give any information on the change of the average kinetic energy in the new orbit. Such an increment, carrying the particle further away from

the centre, is indeed, as a rule, accompanied by a *diminution* of the average energy of motion.

There is a small increase of internal motion due to the magnetic force, which is $ec^{-1}\mathrm{H}v\sin\theta$, and as H is numerically equal to E, the impulse due to the force is $vc^{-1}\sin\theta$ times that due to the electric impulses. We may therefore disregard it, as we retain in each case only the first term of the forces. The action of the magnetic force becomes important in connexion with the translatory motion, because it is there the dominating factor. Our expression for the rate of increase of this kinetic energy remains as before: $\mathrm{NE}^2e^2\tau^2(v\sin\theta)^2/2\mathrm{M}c^2$. Although the above equations still rest on the assumption that the effect of the radiation is concentrated in a series of impulses, I do not think that this assumption need be retained Whatever may be the distribution of the intensities when analysed by Fourier's theorem, it is difficult to resist the conclusion that the electric and magnetic forces affecting the internal and translatory motion respectively are as E to $\mathrm{H}c^{-1}v\sin\theta$; E and H being numerically equal. The corresponding increase of energy must be, in the long run, proportional to the time, and, the angle between the magnetic forces and the direction of velocity of the electron being arbitrary, we may substitute $\frac{2}{3}v^2$ for the average square of $v\sin\theta$. We may, therefore, write for the rate of increase of the internal energy $\mu e^2/m$, and for that of the translatory energy $2\mu e^2v^2/3\mathrm{M}c^2$. Without entering into the question of the meaning of μ, it seems to me that these relations are independent of any particular manner of resolving the radiation into its components. We may now turn to the retarding force. If K be that force, the rate of diminution of energy is $\mathrm{K}v$. But, in the case of electrons, the force is not necessarily directly opposed to the velocity, and we must, therefore, use the scalar product $(\mathrm{K}v)$, or the velocity multiplied by the component of force parallel to it; the other component, being at right angles to the velocity, does no work. Taking the most important term we find in this way for the rate of diminution of internal energy: $2e^2(vv)/3c^3$. When there is a balance between the rates of gain and loss of energy:

$$m\,(v\dot{v}) = \tfrac{3}{2}\mu c^3. \tag{8}$$

To obtain the force retarding the translatory velocity, I must refer to the equations of Abraham.* We replace the velocity of the electron by its two parts u and v, the former relating to the translatory motion. We may then reject the periodic terms. The forces in that case, omitting terms of the second order, reduce to $2e^2u(v\ddot{v})/3c^5$. Again balancing the gain and loss of energy, we find:

$$\mathrm{M}\,(v\ddot{v})\,u^2 = \mu c^3v^3. \tag{9}$$

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* *Loc. cit.*, p. 123.

Equations (8) and (9) lead to

$$\mathrm{M}u^2 = \tfrac{2}{3}mv^2. \tag{10}$$

This is in agreement with (5), where Q represents the sum of the average kinetic and potential energies when the two are equal. But in the general case the relation between the total internal energy and the kinetic energy depends on the law of force. The relation has been worked out by Clausius,* who shows, *e.g.*, that if the law be that of the inverse square, impulses like those here considered would, while increasing the total energy of the internal motion, actually diminish its kinetic energy by an equal amount. The absorption of light would result mainly in an increase of potential energy equal to twice the work done, while the internal kinetic energy and the translatory energy would diminish in the same proportion as the absorbed energy. It appears, therefore, that the distribution of energy between the internal and external energy depends on the forces regulating the internal motion, but no reasonable law for these forces can be given that would lead to an exact equi-partition.

10. It remains to prove equation (2) referred to in § 4, for a fraction of the total energy of the light impulses, which we may consider to be contained within a definite range of freqnency. If a vector $f(t)$ defining the radiation be introduced such that the energy per unit volume is proportional to $[(f(ct)]^2$, we may express $f(t)$ in terms of Fourier's series. The integral $\int_{-\infty}^{+\infty} f(\lambda)\cos\omega\lambda\,d\lambda$ and the corresponding sine integral will, in the case of a beam of light of infinite extent, become infinite, but we avoid the consequent difficulty by taking the limits of the integrals to be finite, though indefinitely large. Between the limits $\pm$T, we now have

$$\pi f(t) = \int_0^\infty d\omega \int_{-\mathrm{T}}^{+\mathrm{T}} f(\lambda)\cos\omega(\lambda - t)\,d\lambda,$$

we derive a second integral† giving the total energy of radiation

$$\pi\int_{-\mathrm{T}}^{+\mathrm{T}} [f(t)]^2\,dt = \int_0^\infty (\mathrm{A}^2+\mathrm{B}^2)\,d\omega,$$

where

$$\mathrm{A} = \int_{-\mathrm{T}}^{+\mathrm{T}} f(\lambda)\cos\omega\lambda\,d\lambda, \qquad \mathrm{B} = \int_{-\mathrm{T}}^{+\mathrm{T}} f(\lambda)\sin\omega\lambda\,d\lambda.$$

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* 'Phil. Mag.,' vol. 42, p. 327 (1847).

† Rayleigh, 'Phil. Mag.,' vol. 27, p. 460 (1889); Schuster, 'Phil. Mag.,' vol. 37, p. 533 (1894).

The average value of $[f(t)]^2$ during the time 2T is therefore

$$(2\pi T)^{-1}\int_0^\infty (A^2+B^2)\,d\omega.$$

Though A and B are infinite, A^2 and B^2 increase on the average in proportion to the time, and the energy transmitted per unit range of frequency $\omega/2\pi$ soon converges with increasing T to a definite limit. Applying the result to the case where $f(t)$ has a constant value E during an interval τ at time t, and zero values at all other times, we find:

$$A = 2E\omega^{-1}\cos\omega t\sin\tfrac{1}{2}\omega\tau, \qquad B = -2E\omega^{-1}\sin\omega t\sin\tfrac{1}{2}\omega\tau.$$

$$A^2+B^2 = 4E^2\omega^{-2}\sin^2\tfrac{1}{2}\omega\tau.$$

If a number of equal impulses follow each other at irregular intervals, A and B will depend on T, but A^2+B^2 has the same value for each impulse. We may therefore obtain the average energy by multiplying the expression by the total number of impulses, of which we shall suppose there are N per unit time. We then obtain for the average value of $[f(t)]^2$

$$\pi^{-1}NE^2\int_0^\infty 4\omega^{-2}\sin^2\tfrac{1}{2}\omega\tau\,d\omega.$$

To calculate the energy per unit volume we must apply the factor $(4\pi)^{-1}$, and for small values of τ we then find the energy-density per unit range of frequency to be $NE^2\tau^2/2\pi$.